Synchronization in weighted uncorrelated complex networks in a noisy environment: Optimization and connections with transport efficiency

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Motivated by synchronization problems in noisy environments, we study the Edwards-Wilkinson process on weighted uncorrelated scale-free networks. We consider a specific form of the weights, where the strength (and the associated cost) of a link is proportional to $(k_i k_j)^\beta$ with k_i and k_j being the degrees of the nodes connected by the link. Subject to the constraint that the total edge cost is fixed, we find that in the mean-field approximation on uncorrelated scale-free graphs, synchronization is optimal at $\beta^*=-1$. Numerical results, based on exact numerical diagonalization of the corresponding network Laplacian, confirm the mean-field results, with small corrections to the optimal value of β^*. Employing our recent connections between the Edwards-Wilkinson process and resistor networks, and some well-known connections between random walks and resistor networks, we also pursue a naturally related problem of optimizing performance in queue-limited communication networks utilizing local weighted routing schemes.

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I. INTRODUCTION

Synchronization in natural and artificial complex interconnected systems [1–4] has been the focus of interdisciplinary research with applications ranging from neurobiology [5,6], ecology and population dynamics [7,8] to scalable computer networks [9–13]. In the recent flood of research on complex networks [14–19], the focus has shifted from structure to various dynamical and stochastic processes on networks, synchronization and transport are being one of them. The common question addressed by most studies within their specific context is how the collective response of locally-coupled entities is influenced by the underlying network topology.

A large number of studies investigated the Kuramoto model of coupled oscillators [4,20], naturally generalized to complex networks [21–23]. The common feature of the findings is the spontaneous emergence of order (synchronous phase) on complex networks, qualitatively similar to that observed on fully-connected networks (also referred to as complete graphs), in contrast to regular networks in low dimensions. Another large group of studies addressed synchronization in coupled nonlinear dynamical systems (e.g., chaotic oscillators) [3] on small-world (SW) [24] and scale-free (SF) [25–29] networks. The analysis of synchronization in the latter models can be carried out by linearization about the synchronous state and using the framework of the master stability function [30]. In turn, the technical challenge of the problem is reduced to the diagonalization of the Laplacian on the respective network, and calculating or estimating the eigenratio [24] (the ratio of the largest and the smallest nonzero eigenvalue of the network Laplacian), a characteristic measure of synchronizability (smaller eigenratios imply better synchronizability). Along these lines, most recent studies [26–28,31] considered not only complex, possibly heterogeneous, interaction topologies between the nodes, but also heterogeneities in the strength of the couplings (also referred to as weighted complex networks).

In a more general setting of synchronization problems, the collective behavior or response of the system is obviously strongly influenced by the nonlinearities, the coupling or interaction topology, the weights or strength of the (possibly directed) links, and the presence and the type of noise [3,29]. Here, we study synchronization in weighted complex networks with linear coupling in the presence of delta-correlated white noise. Although it may appear somewhat simplistic (and, indeed prototypical), such problems are motivated by the dynamics and fluctuations in task-completion landscapes in causally-constrained queuing networks [32,33], with applications in manufacturing supply chains, e-commerce-based services facilitated by interconnected servers [34], and certain distributed-computing schemes on computer networks [9–13]. This simplified problem is the Edwards-Wilkinson (EW) process [35] on the respective network [36–41].

Here, we consider a specific and symmetric form of the weights on uncorrelated SF networks, being proportional to $(k_i k_j)^\beta$ where k_i and k_j are the degrees of the nodes connected by the link [28]. The above general form has been suggested by empirical studies of metabolic [42] and airline transportation networks [43]. Here, we study the effect of such a weighting scheme in our synchronization problem. Associating the weight or strength of each link with its cost, we ask what is the optimal allocation of the weights (controlled by β) in strongly heterogeneous networks, with a fixed total cost, in order to maximize synchronization in a noisy environment. For the EW process on any network, the natural observable is the width or spread of the synchronization landscape [36,38–41]. Then the task becomes minimizing the width as a function of β subject to a (cost) constraint.

Despite differences in the assumptions concerning noise and constrained cost, our results are very similar to the findings by Zhou *et al.* [28], who investigated synchronization of

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coupled nonlinear oscillators on the same type of network. The optimal value of β is close to -1 (and is exactly -1 in the mean-field approximation on uncorrelated random SF networks). The two problems are inherently related through the eigenvalue spectrum of the same network Laplacian.

Transport and flow on complex networks have also become the subject of intensive research with applications to biological, transportation, communication, and infrastructure networks [31,43–67]. While our primary motivation is the above described synchronization phenomena in noisy environments, we also explore some natural connections with idealized transport and flow problems on complex networks, in particular, connections with local routing schemes [57,58,62]. Connections between random walks and resistor networks have been discussed in detail in several works [68–70]. Further, we have recently pointed out [36] some useful connections between the EW process and resistor networks (both systems' behavior is governed by the same network Laplacian). Thus our results for the synchronization problem have some straightforward implications on the related resistor network and random walk problems, pursued in the second part of this work.

The remainder of the paper is organized as follows. In Sec. II, we present results for the EW synchronization problem on weighted uncorrelated SF networks from a constrained optimization viewpoint. In Sec. III, we discuss the related questions for idealized transport problems: weighted resistor networks and weighted random walks. A brief summary is given in Sec. IV.

II. OPTIMIZATION OF SYNCHRONIZATION IN WEIGHTED COMPLEX NETWORKS IN NOISY ENVIRONMENTS

The EW process on a network is given by the Langevin equation

$$\partial_t h_i = -\sum_{j=1}^{N} C_{ij}(h_i - h_j) + \eta_i(t), \tag{1}$$

where $h_i(t)$ is the general stochastic field variable on a node (such as fluctuations in the task-completion landscape in certain distributed parallel schemes on computer networks [10,38,39]) and $\eta_i(t)$ is a delta-correlated noise with zero mean and variance $\langle \eta_i(t)\eta_j(t')\rangle = 2\delta_{ij}\delta(t-t')$. Here, $C_{ij}=C_{ji} > 0$ is the symmetric coupling strength between the nodes i and j ($C_{ii}\equiv 0$). Defining the network Laplacian,

$$\Gamma_{ij} \equiv \delta_{ij}C_i - C_{ij}, \tag{2}$$

where $C_i \equiv \Sigma_l C_{il}$, we can rewrite Eq. (1)

$$\partial_t h_i = -\sum_{j=1}^{N} \Gamma_{ij}h_j + \eta_i(t). \tag{3}$$

For the steady-state equal-time two-point correlation function one finds

$$G_{ij} \equiv \langle (h_i - \bar{h})(h_j - \bar{h})\rangle = \hat{\Gamma}^{-1}_{ij} = \sum_{k=1}^{N-1} \frac{1}{\lambda_k}\psi_{ki}\psi_{kj}, \tag{4}$$

where $\bar{h}=(1/N)\Sigma_{i=1}^{N}h_i$ and $\langle\cdots\rangle$ denotes an ensemble average over the noise in Eq. (3). Here, $\hat{\Gamma}^{-1}$ denotes the inverse of Γ in the space orthogonal to the zero mode. Also, $\{\psi_{ki}\}_{i=1}^{N}$ and λ_k, $k=0,1,\ldots,N-1$, denote the kth normalized eigenvectors and the corresponding eigenvalues, respectively. The $k=0$ index is reserved for the zero mode of the Laplacian on the network: All components of this eigenvector are identical and $\lambda_0=0$. The last form in Eq. (4) (the spectral decomposition of $\hat{\Gamma}^{-1}$) can be used to directly employ the results of exact numerical diagonalization. The average steady-state spread or width in the synchronization landscape can be written as [36,38,39]

$$\langle w^2\rangle \equiv \left\langle \frac{1}{N}\sum_{i=1}^{N}(h_i - \bar{h})^2 \right\rangle = \frac{1}{N}\sum_{i=1}^{N} G_{ii} = \frac{1}{N}\sum_{k=1}^{N-1}\frac{1}{\lambda_k}. \tag{5}$$

The above observable is typically self-averaging (confirmed by numerics), i.e., the width $\langle w^2\rangle$ for a sufficiently large, single network realization approaches the width averaged over the network ensemble.

The focus of this section is to optimize synchronization (i.e., minimize the width) on (i) weighted uncorrelated networks with SF degree distribution and (ii) subject to fixed a cost. In the context of this work, we define the total cost C_{tot} simply to equal to the sum of weights over all edges in the network

$$\sum_{i<j} C_{ij} = \frac{1}{2}\sum_{i,j} C_{ij} = C_{\text{tot}}. \tag{6}$$

The elements of the coupling matrix C_{ij} can be expressed in terms of the network's adjacency matrix A_{ij} and the assigned weights W_{ij} connecting node i and j as $C_{ij}=W_{ij}A_{ij}$. Here, we consider networks where the weights are symmetric and proportional to a power of the degrees of the two nodes connected by the link, $W_{ij}\propto(k_ik_j)^{\beta}$. We choose our cost constraint to be such that it is equal to that of the unweighted network, where each link is of unit strength

$$\sum_{i,j} C_{ij} = 2C_{\text{tot}} = \sum_{i,j} A_{ij} = N\bar{k}, \tag{7}$$

where $\bar{k}=\Sigma_i k_i/N=\Sigma_{i,j}A_{ij}/N$ is the mean degree of the graph, i.e., the average cost per edge is fixed. Thus, the question we ask, is how to allocate the strength of the links in networks with heterogeneous degree distributions with fixed total cost in order to optimize synchronization. That is, the task is to determine the value of β which minimizes the width Eq. (5), subject to the constraint Eq. (7).

Combining the form of the weights, $W_{ij}\propto(k_ik_j)^{\beta}$, and the constraint Eq. (7) one can immediately write for the coupling strength between nodes i and j

$$C_{ij} = N\bar{k}\frac{A_{ij}(k_i k_j)^{\beta}}{\sum_{l,n} A_{ln}(k_l k_n)^{\beta}}. \tag{8}$$

From the above it is clear that the distribution of the weights is controlled by a single parameter β, while the total cost is fixed, $C_{\text{tot}}=N\bar{k}/2$.

A. Globally optimal network with fixed cost

Before tackling the above optimization problem for the restricted set of heterogeneous networks and the specific form of weights, one may ask what is the optimum among all networks with fixed cost, for which the EW synchronization problem yields the minimum width. This will serve as a "baseline" reference for our problem. From the above framework it follows that

$$2C_{\text{tot}} = \sum_{i,j} C_{ij} = \sum_i C_i = \sum_i \Gamma_{ii} = \text{Tr}(\Gamma) = \sum_{l\neq 0} \lambda_l. \tag{9}$$

Thus the global optimization problem can be cast as

$$\langle w^2\rangle = \frac{1}{N}\sum_{l=1}^{N-1}\frac{1}{\lambda_l} = \text{minimum}, \tag{10}$$

with the constraint

$$\sum_{l=1}^{N-1} \lambda_l = 2C_{\text{tot}} = \text{fixed}. \tag{11}$$

This elementary extremum problem, Eqs. (10) and (11), immediately yields a solution where all $N-1$ nonzero eigenvalues are equal,

$$\lambda_l = \frac{2C_{\text{tot}}}{N-1}, \quad l = 1,2,\ldots,N-1, \tag{12}$$

and the corresponding *absolute* minimum of the width is

$$\langle w^2\rangle_{\min} = \frac{(N-1)^2}{2NC_{\text{tot}}}. \tag{13}$$

As one can easily see, the above set of identical eigenvalues corresponds to a coupling matrix and network structure where each node is connected to all others with identical strength $C_{ij}=2C_{\text{tot}}/[N(N-1)]$. That is, for fixed cost, the *fully-connected* (FC) network is optimal, yielding the absolute minimum width.

If one now considers the synchronization problem on any network with N nodes, with average degree $\bar{k}$ and with total cost $C_{\text{tot}}=N\bar{k}/2$ to be optimized in some fashion [e.g., with respect to a single parameter β, Eq. (8)], the above result provides an absolute lower bound for the optimal width

$$\langle w^2(\beta)\rangle_{\min} \geq \frac{(N-1)^2}{N^2}\frac{1}{\bar{k}} \simeq \frac{1}{\bar{k}}. \tag{14}$$

The above result is only of our interest in that it provides a *mathematical* absolute upper bound for the synchronization efficiency (absolute lower bound for the width). Also, in light of the mean-field approximation presented in the next section (Sec. II B) one can immediately see that there is a trivial (yet still sparse) network, which systematically approaches the above globally optimal behavior (for sufficiently large k): A perfectly homogeneous random graph where each node is connected to *exactly* k others. For such graphs, for $1 \ll k \ll N$, the width approaches (the optimal) $\langle w^2\rangle \simeq 1/k$.

The subject of this paper, however, is the scenario where the network structure is heterogeneous and *fixed*. Thus the question we ask is the following: What is the best one can do to optimize synchronization in a noisy environment in strongly heterogeneous (scale-free) graphs if one *cannot* choose or change the structure, but can allocate weights (coupling strength) of a simple form and with a fixed total cost.

B. Mean-field approximation on uncorrelated SF networks

First, we approximate the equations of motion [Eq. (1)] by replacing the local weighted average field $(1/C_i)\Sigma_j C_{ij}h_j$ with the global average $\bar{h}$ (the mean-height)

$$\begin{aligned}\partial_t h_i &= -\sum_{j=1}^{N} C_{ij}(h_i - h_j) + \eta_i(t)\\ &= -C_i\left(h_i - \frac{\sum_j C_{ij}h_j}{C_i}\right) + \eta_i(t)\\ &\approx -C_i(h_i - \bar{h}) + \eta_i(t).\end{aligned} \tag{15}$$

As can be directly seen by summing up Eq. (1) over all nodes, the mean height $\bar{h}$ performs a simple random walk with noise intensity $\mathcal{O}(1/N)$. Thus, in the mean-field (MF) approximation, in the asymptotic large-N limit, fluctuations *about the mean* are decoupled and reach a stationary distribution with variance

$$\langle (h_i - \bar{h})^2\rangle \approx 1/C_i, \tag{16}$$

yielding

$$\langle w^2\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle (h_i-\bar{h})^2\rangle \approx \frac{1}{N}\sum_i \frac{1}{C_i}. \tag{17}$$

Next, we establish an approximate relationship between the effective coupling to the mean, C_i, and the degree k_i of node i, for *uncorrelated* (UC) weighted random graphs. Using the specific form of the weights as constructed in Eq. (8), we write

$$C_i = \sum_j C_{ij} = N\bar{k}\frac{\sum_j A_{ij}(k_i k_j)^{\beta}}{\sum_{l,n} A_{ln}(k_l k_n)^{\beta}} = N\bar{k}\frac{k_i^{\beta}\sum_j A_{ij}k_j^{\beta}}{\sum_l k_l^{\beta}\sum_n A_{ln}k_n^{\beta}}. \tag{18}$$

For large minimum (and in turn, average) degree, expressions of the form $\Sigma_j A_{ij} k_j^{\beta}$ can be approximated as

$$\sum_j A_{ij}k_j^\beta = \left(\sum_j A_{ij}\right)\frac{\sum_j A_{ij}k_j^\beta}{\sum_j A_{ij}} = k_i\frac{\sum_j A_{ij}k_j^\beta}{\sum_j A_{ij}} \approx k_i \int dk\, P(k|k_i)k^\beta, \tag{19}$$

where $P(k|k')$ is the probability that an edge from node with degree k' connects to a node with degree k. For *uncorrelated* random graphs, $P(k|k')$ does *not* depend on k', and one has $P(k|k')=kP(k)/\langle k\rangle$ [17,75], where $P(k)$ is the degree distribution and $\langle k\rangle$ is the ensemble-averaged degree. Thus, Eq. (18), for UC random networks, can be approximated as

$$C_i \approx N\langle k\rangle \frac{k_i^{\beta+1}\int dk\, P(k|k_i)k^\beta}{N\int dk' k'^{\beta+1}P(k')\int dk\, P(k|k')k^\beta} = \langle k\rangle \frac{k_i^{\beta+1}}{\int_m^\infty dk' k'^{\beta+1}P(k')}. \tag{20}$$

Here, we consider SF degree distributions,

$$P(k) = (\gamma-1)m^{\gamma-1}k^{-\gamma}, \tag{21}$$

where m is the minimum degree in the network and $2<\gamma\leq 3$. The average and the minimum degree are related through $\langle k\rangle = m(\gamma-1)/(\gamma-2)$. No upper cutoff is needed for the convergence of the integral in Eq. (20), provided that $2+\beta-\gamma<0$, and one finds

$$C_i \approx \frac{\gamma-2-\beta}{\gamma-2}\frac{k_i^{\beta+1}}{m^\beta}. \tag{22}$$

Thus, for uncorrelated random SF graphs with large minimum degree, the effective coupling coefficient C_i only depends on the degree k_i of node i, i.e., for a node with degree k

$$C(k) \approx \frac{\gamma-2-\beta}{\gamma-2}\frac{k^{\beta+1}}{m^\beta}. \tag{23}$$

Finally, assuming self-averaging for large enough networks and combining the above, one obtains for the width of the synchronization landscape

$$\langle w^2(\beta)\rangle \approx \frac{1}{N}\sum_i \frac{1}{C_i} \approx \int_m^\infty dk\, P(k)\frac{1}{C(k)} = \frac{1}{\langle k\rangle}\frac{(\gamma-1)^2}{(\gamma-2-\beta)(\gamma+\beta)}, \tag{24}$$

where using infinity as the upper limit is justified for $\gamma+\beta>0$. Elementary analysis yields the main features of the above expression for the average width as follows.

(1) $\langle w^2(\beta)\rangle$ is minimum at $\beta=\beta^*=-1$, *independent* of the value of γ.

(2) $\langle w^2\rangle_{\rm min} = \langle w^2(\beta^*)\rangle = 1/\langle k\rangle$.

The above approximate result is consistent with using infinity as the upper limit in all integrals, in that the optimal value $\beta^*=-1$ falls inside the interval $-\gamma<\beta<\gamma-2$ for $2<\gamma\leq 3$. Interestingly, one can also observe, that, in this approximation, the minimal value of the width is equal to that of the global optimum [Eq. (14)], realized by the fully connected network of the same cost $N\langle k\rangle/2$, i.e., with identical links of strength $\langle k\rangle/(N-1)$.

We emphasize that in obtaining the above result [Eq. (24)] we employed two very strong and distinct approximations: (i) For the dynamics on the network, we neglected correlations (in a MF fashion) between the local field variables and approximated the local height fluctuations by Eq. (16); (ii) We assumed that the network has no degree-degree correlations between nodes which are connected (UC), so that the "weighted degree" C_i can be approximated with Eq. (22) for networks with $m\gg 1$.

Finally, we note that the average width, in principle, can also be obtained by employing the density of states (DOS) $\rho(\lambda)$ of the underlying weighted network Laplacian through $\langle w^2\rangle = (1/N)\sum_{l=1}^{N-1} 1/\lambda_l \simeq \int (1/\lambda)\rho(\lambda)d\lambda$, in the asymptotic large-N limit [37,38]. Obtaining the DOS analytically, however, is a rather challenging task. Just recently, using the replica method [71,72], Kim and Kahng obtained the DOS for the Laplacian of unweighted ($\beta=0$) SF graphs [73], which they were able to evaluate in the asymptotic $1\ll\langle k\rangle\ll N$ limit. Utilizing their result, we have checked and found full agreement for the width with Eq. (24) for $\beta=0$. Approximate results from the replica-obtained DOS in the $1\ll\langle k\rangle\ll N$ limit, are in essence, of mean-field like, and one expects to find agreement with our generic result Eq. (24) for arbitrary β.

C. Numerical results

For comparison with the above mean-field results, we considered Barabási-Albert (BA) SF networks [14,16], "grown" to N nodes [74], where $P(k)=2m^2/k^3$, i.e., $\gamma=3$. While growing networks, in general, are not uncorrelated, degree-degree correlations are anomalously (marginally) weak for the BA network [75,76].

We have performed exact numerical diagonalization and employed Eq. (4) to find the local height fluctuations and Eq. (5) to obtain the width for a given network realization. We carried out the above procedure for 10–100 independent network realizations. Finite-size effects (except for the $m=1$ BA tree network) are very weak for $-2<\beta<1$; the width essentially becomes independent of the system size. Figure 1 displays result for the local height fluctuations as a function of the degree of the node. We show both the fluctuations averaged over all nodes with degree k and the scattered data for individual nodes. One can observe that our approximate results for the scaling with the degree [combining Eqs. (16) and (22)], $\langle(h_i-\bar{h})^2\rangle \approx 1/C_i \sim k_i^{-(\beta+1)}$, work very well, except

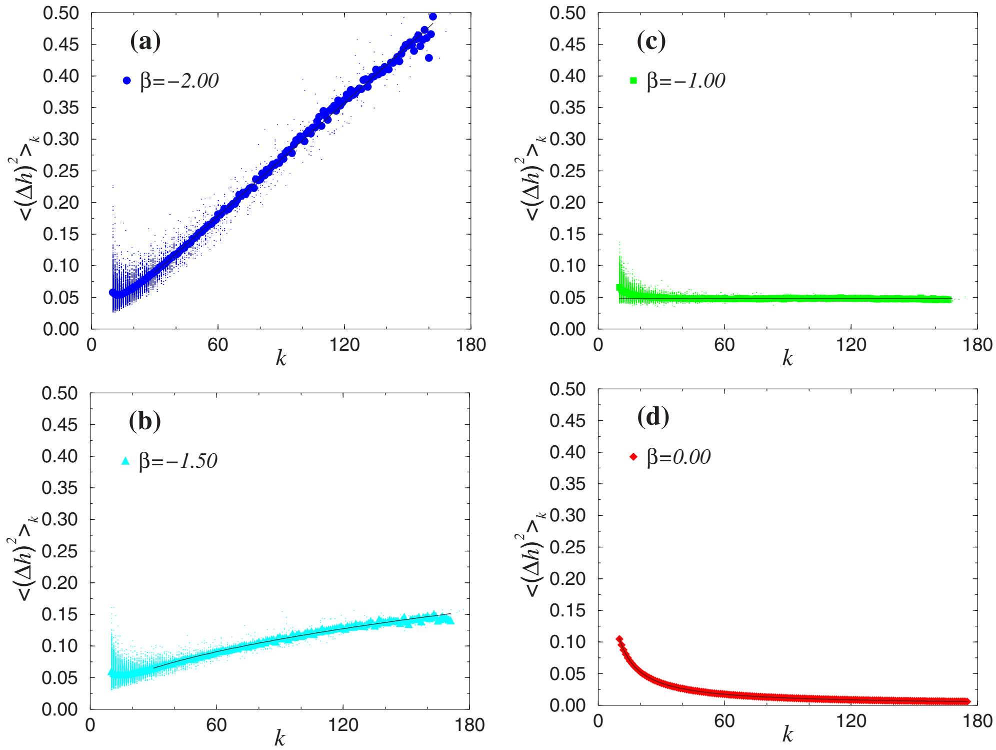

FIG. 1. (Color online) Height fluctuations as a function of the degree of the nodes for $N=1000$, $\langle k\rangle=20$, and for (a) $\beta=-2.00$, (b) $\beta=-1.50$, (c) $\beta=-1.00$, and (d) $\beta=0.00$. Data, represented by filled symbols, are averaged over all nodes with degree k. Scatter plot (dots) for individual nodes is also shown from ten network realizations. Solid lines correspond to the MF+UC scaling $\langle(\Delta h)^2\rangle_k \sim k^{-(\beta+1)}$.

for very low degrees. The special case $\beta=0$, is exceptionally good, since here $C_i=\Sigma_j A_{ij}=k_i$ exactly, and the only approximation is Eq. (16).

In Fig. 2, we show our numerical results for the width and compare it with the approximate (MF+UC) results Eq. (24). They agree reasonably well for networks with $m\gg 1$. The divergence of the approximate result Eq. (24) at $\beta=-3$ and $\beta=1$ is the artifact of using infinity as the upper limit in the integrals performed in our approximations.

The results for the width clearly indicate the existence of a minimum at a value of β^* somewhat greater than -1. As the minimum degree m is increased, the optimal β approaches -1 from above. This is not surprising, since in the limit of $m\gg 1$ (large minimum degree), both the MF and the UC part of our approximations are expected to work progressively better. In Fig. 3, we show the width as a function of $1/m$ for the BA networks, indicating the rate of convergence to the MF+UC result, Eq. (24). Figure 3 also indicates that finite-size effects are very small and only contribute as small corrections to the *finite* value of the width in the limit of $N\to\infty$. For $\beta=0$, our approximation [Eq. (24)] is within 8%, 4%, and 1% of the results extracted from exact numerical diagonalization through Eq. (5), for $m=10$, $m=20$, and $m=100$, respectively. For $\beta=-1$, it is within 15%, 7%, and 3% of the numerical results for $m=10$, $m=20$, and $m=100$, respectively. Thus our approximation works very well for large uncorrelated sparse SF networks with sufficiently large minimum (and consequently, average) degree.

The above optimal link-strength allocation at around the value $\beta^*=-1$ seems to be present in all random networks where the degree distribution is different from a delta function. For example, in SW networks [77–79], although the degree distribution has an exponential tail, $\langle w^2\rangle$ also exhibits a minimum, but the effect is much weaker, as shown in Fig. 2(a). Further, a point worthwhile to mention, a SW network with the same number of nodes and the same average degree (corresponding to the same cost) always "outperforms" its SF counterpart (in terms of minimizing the width). The difference between their performance is smallest around the optimal value, where both are very close to that of the lowest possible value, realized by the FC network of the same cost (Table I).

III. CONNECTIONS WITH TRANSPORT AND FLOW PROBLEMS IN WEIGHTED NETWORKS

A. Optimizing the system resistance in weighted resistor networks

Resistor networks have been widely studied since the 1970s as models for conductivity problems and classical transport in disordered media [80,81]. Amidst the emerging research on complex networks, resistor networks have been employed to study and explore community structures in social networks [82–85]. Also, electrical networks with directed links (corresponding to diodes) have been used to propose novel page-ranking methods for search engines on the World-Wide-Web [86].

Most recently, simple resistor networks were utilized to study transport efficiency in SF [64,65] and SW networks

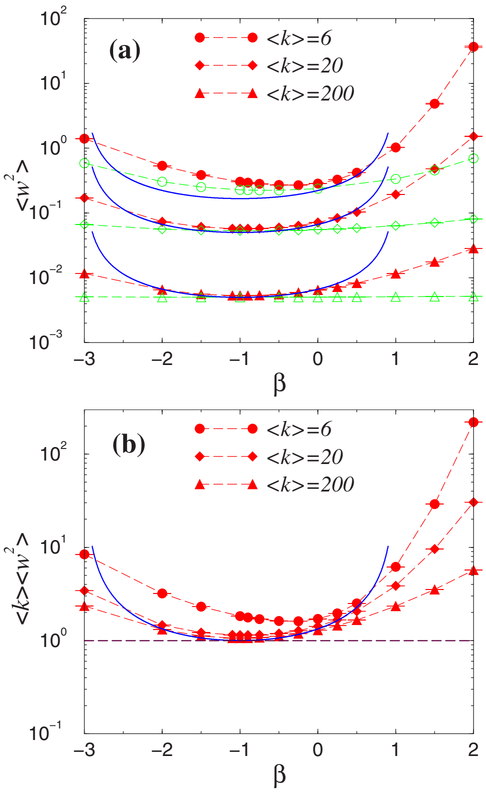

FIG. 2. (Color online) (a) Steady-state width of the EW synchronization landscape as a function of the weighting parameter β for the BA networks for $N=1000$ with various average degree $\bar{k} \simeq \langle k\rangle \simeq 2m$. Solid curves are the approximate (MF+UC) results [Eq. (24)] for the same degree. For comparison, numerical results for SW networks with the same degree (with the respective open symbols) are also shown. Also, see Table I for actual numerical values for $\beta=-1$. (b) Scaled width as a function of the weighting parameter β. The solid curve is the scaled approximate (MF+UC) result [Eq. (24)]. The horizontal dashed line indicates the (similarly scaled) absolute lower bound, as achieved by the fully connected network with the same cost $N\langle k\rangle/2$.

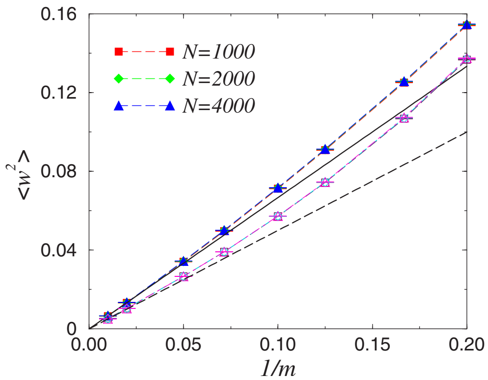

FIG. 3. (Color online) Steady-state width of the EW synchronization landscape as a function of $1/m$ for the BA networks for $\beta=0.00$ (solid symbols) and $\beta=-1.00$ (respective open symbols), for three different system sizes. Straight lines (solid for $\beta=0$ and dashed for $\beta=-1$) correspond to the MF+UC approximation Eq. (24).

[36]. The work by López *et al.* [65] revealed that in SF networks [14,16] anomalous transport properties can emerge, displayed by the power-law tail of distribution of the network conductance. Now, we consider weighted resistor networks subject to a fixed total cost (the cost of each link is associated with its conductance).

In a recent paper we have shown that observables in the EW synchronization problem and in (ohmic) resistor networks are inherently related through the spectrum of the network Laplacian [36]. Consider an arbitrary (connected) network where C_{ij} is the conductance of the link between node i and j, with a current I entering (leaving) the network at node s (t). Introducing the voltages measured from the mean at each node, $\hat{V}_i=V_i-\bar{V}$, where $\bar{V}=(1/N)\Sigma_{i=1}^{N}V_i$, one obtains [36]

$$\hat{V}_i = I(G_{is}-G_{it}). \tag{25}$$

Here, G is the same network propagator discussed in the context of the EW process, i.e., the inverse [Eq. (4)] of the network Laplacian [Eq. (2)] in the space orthogonal to the zero mode. Applying Eq. (25) to nodes s and t, where the voltage drop between these nodes is $V_{st}=\hat{V}_s-\hat{V}_t$, one immediately obtains the effective two-point resistance of the network between nodes s and t [36,87],

$$R_{st} \equiv \frac{V_{st}}{I} = G_{ss}+G_{tt}-2G_{st} = \sum_{k=1}^{N-1}\frac{1}{\lambda_k}(\psi_{ks}^2+\psi_{kt}^2-2\psi_{ks}\psi_{kt}). \tag{26}$$

The spectral decomposition in Eq. (26) is, again, useful to employ the results of exact numerical diagonalization. Comparing Eqs. (4) and (26), one can see that the two-point resistance of a network between node s and t is the same as the steady-state *height-difference* correlation function of the EW process on the network [36],

$$\langle(h_s-h_t)^2\rangle = \langle[(h_s-\bar{h})-(h_t-\bar{h})]^2\rangle = G_{ss}+G_{tt}-2G_{st}=R_{st}. \tag{27}$$

For example, using the above relationship and employing the MF+UC approximation [88], one can immediately obtain

TABLE I. Comparing numerical values of the steady-state width $\langle w^2\rangle$ of the EW process at $\beta=-1$ for BA and SW networks of the same finite average degree $\langle k\rangle$ and cost $N\langle k\rangle/2$ (for $N=1000$) with results of the MF+UC approximation Eq. (24). Note that the width in the MF+UC approximation for $\beta=-1$ coincides with that of the globally optimal FC network of the *same cost* [compare Eqs. (14) and (24)], $\langle w^2\rangle \simeq 1/\langle k\rangle$. Error bars on the numerically obtained values for the BA and SW networks are less that the last digit shown in the table.

$\langle k\rangle$	BA	SW	FC
6	0.304	0.228	0.1666
20	0.0571	0.0531	0.0500
200	0.0053	0.00501	0.0050

the scaling of the typical value of the effective two-point resistance in weighted resistance networks, between two nodes with degrees k_s and k_t,

$$R_{st} \simeq G_{ss} + G_{tt} \sim [k_s^{-(1+\beta)} + k_t^{-(1+\beta)}] = \frac{k_s^{1+\beta} + k_t^{1+\beta}}{(k_s k_t)^{1+\beta}}. \quad (28)$$

A global observable, measuring transport efficiency, analogous to the width of the synchronization landscape, is the average two-point resistance [36,65] (averaged over all pairs of nodes, for a given network realization). Using Eq. (27) and exploiting the basic properties of the Green's function, one finds

$$\bar{R} \equiv \frac{2}{N(N-1)} \sum_{s<t} R_{st} = \frac{1}{N(N-1)} \sum_{s \neq t} R_{st} = \frac{N}{N-1} 2\langle w^2 \rangle \simeq 2\langle w^2 \rangle, \quad (29)$$

i.e., in the asymptotic large system-size limit the average system resistance of a given network is twice the steady-state width of the EW process on the same network. Note that the above relationships, Eqs. (27) and (29), are exact and valid for any graph.

The corresponding optimization problem for resistor networks then reads as follows: For a fixed total cost, $C_{\text{tot}} = \Sigma_{i<j} C_{ij} = N\langle k \rangle / 2$, where the link conductances are weighted according to Eq. (8), what is the value of β which minimizes the average system resistance $\bar{R}(\beta)$? Based on the above relationship between the average system resistance and the steady-state width of the EW process on the same graph [Eq. (29)], the answer is the same as was discussed in Sec. II [Eq. (24)]: $\beta^* = -1$ and $\bar{R}_{\min} = 2N/[(N-1)\langle k \rangle] \simeq 2/\langle k \rangle$ in the mean-field approximation on uncorrelated random SF networks. Numerical results for $\bar{R}(\beta)$ are also provided for "free" by virtue of the connection Eq. (29), once we have the results for $\langle w^2(\beta) \rangle$.

B. Connection with random walks on weighted networks and congestion-aware local routing schemes

Consider the weights $\{C_{ij}\}$ employed in the previous sections and define a discrete-time random walk (RW) with the transition probabilities [68]

$$P_{ij} \equiv \frac{C_{ij}}{C_i} \quad (30)$$

and recall that $C_i = \Sigma_l C_{il}$. P_{ij} is the probability that the walker currently at node i will hop to node j in the next step. Note that because of the construction of the transition probabilities (being a normalized ratio), the issue of cost constraint disappears from the problem. That is, any normalization prefactor associated with the conserved cost [as in Eq. (8)] cancels out, and the only relevant information is $C_{ij} \propto A_{ij}(k_i k_j)^\beta$, yielding

$$P_{ij} = \frac{C_{ij}}{C_i} = \frac{A_{ij}(k_i k_j)^\beta}{\sum_l A_{il}(k_i k_l)^\beta} = \frac{A_{ij} k_j^\beta}{\sum_l A_{il} k_l^\beta}. \quad (31)$$

Conversely, the results are invariant for any normalization (constraint), so for convenience, one can use the normalized form of the C_{ij} coefficients as given in Eq. (8).

Having a random walker starting at an arbitrary source node s, tasked to arrive at an arbitrary target node t, the above weighted RW model can be associated with a simple *local* routing or search scheme [57] where packets are independently forwarded to a nearest neighbor, chosen according to the transition probabilities Eq. (31), until the target is reached. These probabilities contain only limited local information, namely the degree of all neighboring nodes. By construction, the associated local (stochastic) routing problem (Sec. III B 3) does not concern link strength (bandwidth) limitations but rather the processing or queuing capabilities of the nodes, so the cost constraint, associated with the links, disappears form the problem.

1. Node betweenness for weighted RWs

In network-based transport or flow problems, the appropriate betweenness measure is defined to capture the amount of traffic or information passing through a node or a link, i.e., the load of a node or a link [17,44–46,75,85,89,90]. Here, our observable of interest is the *node betweenness* B_i for a given routing scheme [57] (here, purely local and characterized by a single parameter β): *The expected number of visits* to node i for a random walker originating at node s (the source) before reaching node t (the target) $E_i^{s,t}$, summed over all source-target pairs. For a general RW, as was shown by Doyle and Snell [68], $E_i^{s,t}$ can be obtained using the framework of the equivalent resistor-network problem (discussed in Sec. III A). More specifically,

$$E_i^{s,t} = C_i(V_i - V_t), \quad (32)$$

while a *unit* current is injected (removed) at the source (target) node. Utilizing again the network propagator and Eq. (25), one obtains

$$E_i^{s,t} = C_i(V_i - V_t) = C_i(\hat{V}_i - \hat{V}_t) = C_i(G_{is} - G_{it} - G_{ts} + G_{tt}). \quad (33)$$

For the node betweenness, one then obtains

$$B_i = \sum_{s \neq t} E_i^{s,t} = \frac{1}{2} \sum_{s \neq t} (E_i^{s,t} + E_i^{t,s}) = \frac{1}{2} \sum_{s \neq t} C_i(G_{ss} + G_{tt} - 2G_{ts}) = \frac{C_i}{2} \sum_{s \neq t} R_{st} = \frac{C_i}{2} N(N-1)\bar{R}. \quad (34)$$

Note that the above expression is valid for any graph and for an arbitrary weighted RW defined by the transition probabilities Eq. (30). As can be seen from Eq. (34), the node betweenness is proportional to the product of a local topological measure, the weighted degree C_i, and a global flow measure, the average system resistance $\bar{R}$. As a specific case, for the unweighted RW ($\beta=0$) $C_i=\sum_l A_{il}=k_i$; thus the node betweenness is exactly proportional to the degree of the node, $B_i=k_i N(N-1)\bar{R}/2$.

Using our earlier approximations and results for uncorrelated SF graphs Eqs. (22) and (24), and the relationship between the width and the average system resistance Eq. (29), for weighted RW, controlled by the exponent β, we find

$$B_i(\beta)=\frac{C_i}{2}N(N-1)\bar{R}=C_i N^2\langle w^2\rangle \approx N^2\frac{\gamma-1}{\gamma+\beta}\frac{k_i^{1+\beta}}{m^{1+\beta}}. \tag{35}$$

First, we consider the average "load" of the network

$$\bar{B}=\frac{1}{N}\sum_i B_i=\frac{\sum_i C_i}{2}(N-1)\bar{R}. \tag{36}$$

Similar to Eq. (34), the above expression establishes an exact relationship between the average node betweenness of an arbitrary RW [given by Eq. (30)] and the observables of the associated resistor network, the total edge cost and the average system resistance. For example, for the $\beta=0$ case, $\bar{B}=\bar{k}N(N-1)\bar{R}/2$. As noted earlier, for calculation purposes one is free to consider the set of C_{ij} coefficients given by Eq. (8), which also leads us to the following statement:

For a RW defined by the transition probabilities Eq. (30), the average RW betweenness is minimal when the average system resistance of the associated resistor network with fixed total edge cost (and the width of the associated noisy synchronization network) is minimal.

Utilizing again our earlier approximations and results for uncorrelated SF graphs and the relationship between the width and the average system resistance, we find

$$\bar{B}(\beta)=\frac{\sum_i C_i}{2}(N-1)\bar{R} =\left(\sum_i C_i\right)N\langle w^2\rangle \approx N^2\frac{(\gamma-1)^2}{(\gamma-2-\beta)(\gamma+\beta)}. \tag{37}$$

The average node betweenness is minimal for $\beta=\beta^*=-1$, for all γ.

2. Commute times and hitting times for weighted RWs

The hitting time τ_{st} is the expected number of steps for the random walker originating at node s to reach node t for the first time. The commute time is the expected number of steps for a "round trip" between nodes s and t, $\tau_{st}+\tau_{ts}$. Relationships between the commute time and the effective two-point resistance have been explored and discussed in detail in several works [69,91,92]. In its most general form, applicable to weighted networks, it was shown by Chandra *et al.* [91] that

$$\tau_{st}+\tau_{ts}=\left(\sum_i C_i\right)R_{st}. \tag{38}$$

For the average hitting (or first passage) time, averaged over all pairs of nodes, one then obtains

$$\bar{\tau}\equiv\frac{1}{N(N-1)}\sum_{s\neq t}\tau_{s,t}=\frac{1}{2N(N-1)}\sum_{s\neq t}(\tau_{s,t}+\tau_{t,s}) =\frac{\sum_i C_i}{2N(N-1)}\sum_{s\neq t}R_{st}=\frac{\sum_i C_i}{2}\bar{R}. \tag{39}$$

Comparing Eqs. (36) and (39), the average hitting time (the average travel time for packets to reach their destinations) then can be written as $\bar{\tau}=\bar{B}/(N-1)$. Note that this relationship is just a specific realization of Little's law [93,94], in the context of general communication networks, stating that the average time needed for a packet to reach its destination is proportional to the total load of the network. Thus, the average hitting time and the average node betweeness (only differing by a factor of $N-1$) are minimized *simultaneously* for the same graph (as a function of β, in our specific problem).

3. Network congestion due to queuing limitations

Consider the simplest local "routing" problem [57,62] in which packets are generated at *identical* rate ϕ at each node. Targets for each newly generated packet are chosen uniformly at random from the remaining $N-1$ nodes. Packets perform independent, weighted RWs, using the transition probabilities Eq. (30), until they reach their targets. Further, the queuing or processing capabilities of the nodes are limited and are identical, e.g. (without loss of generality) each node can send out one packet per unit time. From the above it follows that the network is congestion-free as long as

$$\phi\frac{B_i}{N-1}<1, \tag{40}$$

for *every* node i [56,57,60,61,63]. As the packet creation rate ϕ (network throughput per node) is increased, congestion emerges at a critical value ϕ_c when the inequality in Eq. (40) is first violated. Up to that point, the simple model of independent random walkers (discussed in the previous subsections), can self-consistently describe the average load landscape in the network. Clearly, network throughput is limited by the most congested node (the one with the maximum betweeness), thus

$$\phi_c=\frac{N-1}{B_{\max}}, \tag{41}$$

a standard measure to characterize the efficiency of communication networks [56,57,60,61,63].

To enhance or optimize network throughput (limited by the onset of congestion at the nodes), one may scale up the processing capabilities of the nodes [60], optimize the underlying network topology [57], or optimize routing by finding pathways which minimize congestion [61–63]. The above RW routing, controlled by the weighting parameter β, is an example for the latter, where the task is to maximize global

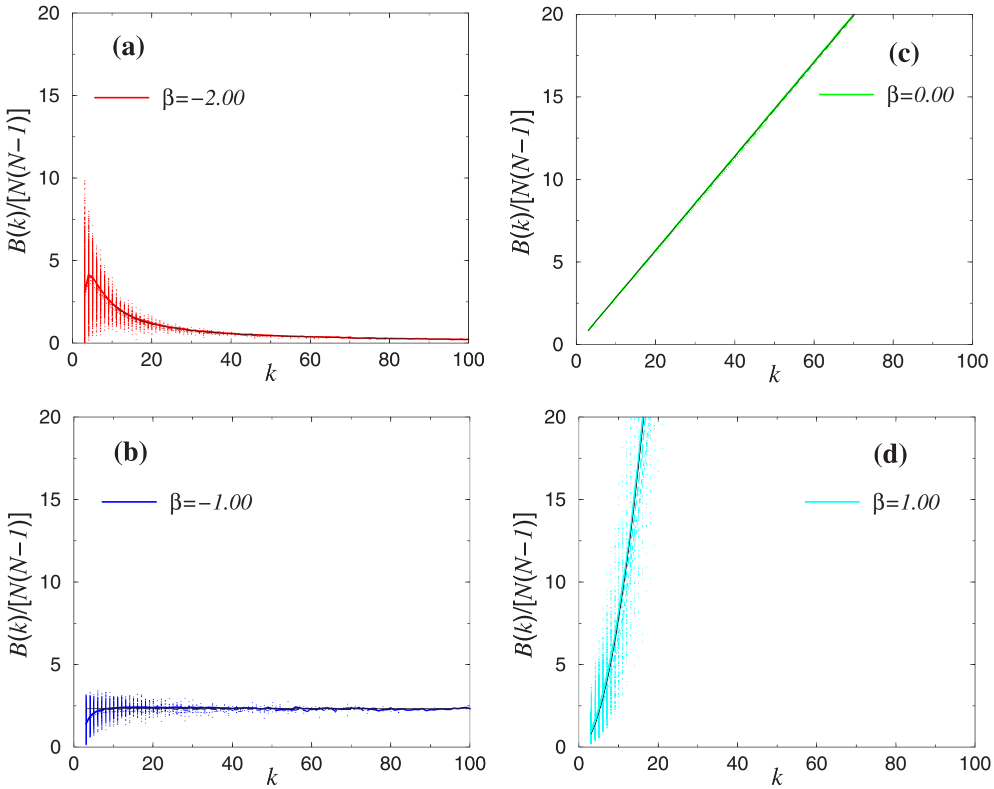

FIG. 4. (Color online) Normalized RW node betweenness on BA networks with $m=3$ as a function of the degree of the nodes for four system sizes [$N=$ 200 (dotted), 400 (dashed), 1000 (long-dashed), 2000 (solid)] for (a) $\beta=-2.00$, (b) $\beta=-1.00$, (c) $\beta=0.00$, and (d) $\beta=1.00$. Data point represented by lines are averaged over all nodes with degree k. Data for different system sizes are essentially indistinguishable. Scatter plot (dots) for the individual nodes is also shown from ten network realizations for $N=1000$. Solid curves, corresponding to the MF+UC scaling $B(k)\sim k^{\beta+1}$ [Eq. (35)], are also shown.

network throughput by locally directing traffic. In general, congestion can also be strongly influenced by "bandwidth" limitations (or collisions of packets), which are related to the edge betweenness, and not considered here.

For $\beta>-1$, within our approximations, nodes with high betweenness coincide with nodes with high degree. Further, for nodes with high degree, the mean-field approach on uncorrelated SF graphs is expected to work reasonably well. In this region, the scaling behavior $B_{\max}$ is related to that of the highest degree $k_{\max}$ in the graph of *finite* size N. The scaling of the maximum degree with the system size, however, even for idealized SF network models, is very sensitive to the details of the network construction. For example, in the region of our interest, $2<\gamma\leq 3$, for the standard configuration model (CM) [95], the largest degree is governed by the *natural cutoff*, $k_{\max}\simeq mN^{1/(\gamma-1)}$ [17,96], but this network has correlations, especially between nodes with larger degrees [76,96]. So one may use the MF+UC approximation, but should expect stronger corrections. On the other hand, in a recent construction for SF networks, the uncorrelated configurational model (UCM) [76], the resulting network is genuinely uncorrelated, and the largest degree is governed by the *structural cutoff*, $k_{\max}\simeq(\langle k\rangle N)^{1/2}$ [76,96–98]. Combining these cutoff behaviors with Eq. (35), for the CM scale-free network model with the natural cutoff one has

$$B_{\max}^{\mathrm{CM}}(\beta)\approx N^2\frac{\gamma-1}{\gamma+\beta}\frac{k_{\max}^{1+\beta}}{m^{1+\beta}}\simeq\frac{\gamma-1}{\gamma+\beta}N^{(2\gamma+\beta-1)/(\gamma-1)} \tag{42}$$

and

$$\phi_c^{\mathrm{CM}}(\beta)=\frac{N-1}{B_{\max}}\approx\frac{\gamma+\beta}{\gamma-1}N^{-(\gamma+\beta)/(\gamma-1)}\sim\mathcal{O}(N^{-(\gamma+\beta)/(\gamma-1)}). \tag{43}$$

Similarly, for the UCM scale-free network model one finds

$$B_{\max}^{\mathrm{UCM}}(\beta)\simeq\frac{\gamma-1}{\gamma+\beta}N^2\left(\frac{\gamma-1}{\gamma-2}\frac{N}{m}\right)^{(1+\beta)/2} \tag{44}$$

and

$$\phi_c^{\mathrm{UCM}}(\beta)\simeq\frac{\gamma+\beta}{\gamma-1}\frac{1}{N}\left(\frac{\gamma-1}{\gamma-2}\frac{N}{m}\right)^{-(1+\beta)/2}\sim\mathcal{O}(N^{-(3+\beta)/2}). \tag{45}$$

From the above expression one can see that in the $\beta>-1$ region, for large N, the exponential decay in β dominates for both the CM [Eq. (43)] and UCM [Eq. (45)] scale-free networks. Consequently, in the semi-infinite region $\beta>-1$, $\phi_c(\beta)$ is a monotonically decreasing function of β.

For $\beta<-1$, nodes with high betweenness are the nodes with a low degree, but for these nodes the above approximations are expected to work poorly. Further, there are many nodes with a degree of order m, and the actual distribution of the betweenness [through the weighted degrees C_i, Eq. (34)] for nodes with $k_i \sim m$, depends strongly on the "local" fluctuations of the network disorder (randomness of the network structure). Ignoring all of the above, and blindly using Eq. (35) with $k_{\min}=m$, one finds $\phi_c(\beta) \approx [(\gamma+\beta)/(\gamma-1)]N^{-1}$, which is a monotonically increasing function of β in the semi-infinite region $\beta<-1$. Thus, within our crude approximate scheme, the throughput is maximum at $\beta^*=-1$.

Numerical work, performed on the BA network ($\gamma=3$), supports the above simple analysis. The BA network is somewhat special, in that correlations are anomalously weak (or marginal), and the structural and natural cutoffs exhibit the *same* $\mathcal{O}(N^{1/2})$ scaling with the system size. Testing our MF+UC predictions, we find that the betweenness is, indeed, strongly correlated with the degree, in line with Eq. (35) (Fig. 4). Further, for $\beta>\beta^* \approx -1$, the tail of the degree distribution governs the tail of the distribution of the betweenness. Specifically, the cumulative degree distribution, $P_>(k) \sim k^{1-\gamma}$ translates to the cumulative betweenness distribution $P_>(B) \sim B^{(1-\gamma)/(1+\beta)}$ (Fig. 5). For $\beta<\beta^* \approx -1$, as noted above, the large-B tail of the betweenness distribution is coming from the small-k behavior of the degree distribution. While there is a strict lower cutoff in the degrees m, there are many nodes with degree m. It is then the quenched randomness in the particular network realization which ultimately governs the upper cutoff of the betweenness (through the weighted degrees C_i). The tail of the betweenness distribution is essentially independent of N and numerically found to scale in an exponential-like fashion (Fig. 5).

As qualitatively predicted by the MF+UC approximation, the critical network throughput $\phi_c(\beta)$ exhibits a maximum at around $\beta^* \approx -1$, corresponding to the optimal weighting scheme, as shown in Figs. 6. Further, in the $\beta>-1$ region, where the long tail of the degree distribution dominates the network behavior, the network throughput scales with the number of nodes as $\sim N^{-(\gamma+\beta)/(\gamma-1)}$. [Note that for the BA network ($\gamma=3$), the scaling with N by Eqs. (43) and (45) coincide.] The results for the scaled throughput are shown in Figs. 7.

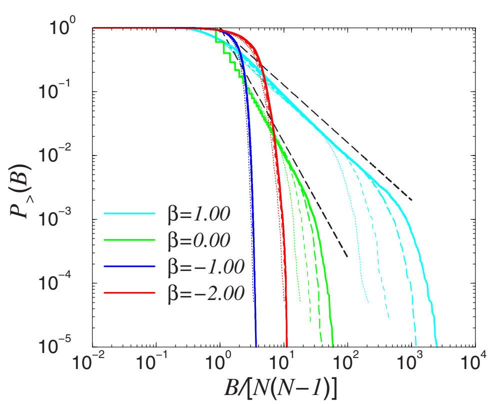

FIG. 5. (Color online) Cumulative distributions of the normalized RW node betweenness for BA networks with $m=3$, for four values of β, each with four system sizes. The four families of cumulative distributions correspond to the four different values of β, from left to right in the figure, $\beta=-1.00$, -2.00, 0.00, and 1.00. Each family has four system sizes: $N=200$ (dotted), 400 (dashed), 1000 (long-dashed), and 2000 (solid curves). Finite-size effects are only significant for $\beta=0.00$ and $\beta=1.00$. Straight dashed lines correspond to the predicted power-law tail of the cumulative distribution for $\beta>-1$, $P_>(B) \sim B^{(1-\gamma)/(1+\beta)}$.

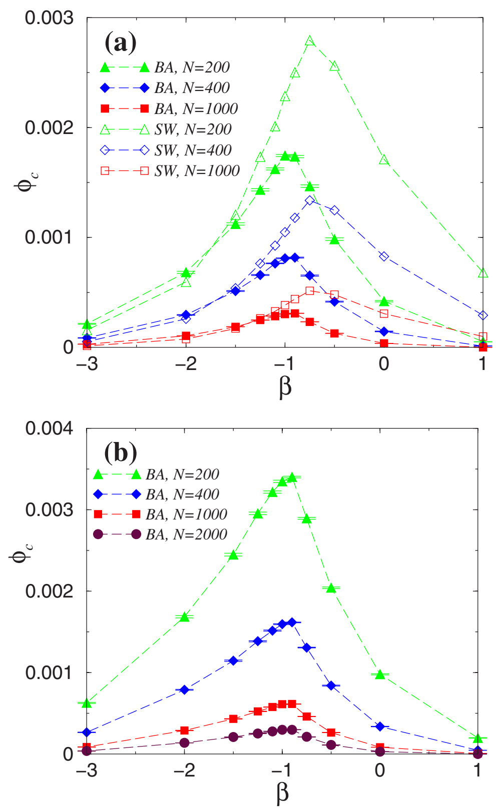

FIG. 6. (Color online) Critical network throughput per node as a function of the weighting parameter β for BA networks (solid symbols) for various system size for (a) $m=3$ and for (b) $m=10$. Figure (a) also shows the same observable for SW networks (the same respective open symbols) for the same system sizes.

In a recent, more realistic network traffic simulation study of a congestion-aware routing scheme, Danila *et al.* [62] found qualitatively very similar behavior to what we have observed here. In their network traffic simulation model, packets are forwarded to a neighbor with a probability proportional to a power β of the *instantaneous queue length* of the neighbor. They found that there is an optimal value of the exponent β, close to -1.

We also show numerical results for the network throughput for SW networks with the same degree [Fig. 6(a)]. In particular, an optimally weighted SW network always outperforms its BA scale-free counterpart with the same degree. Qualitatively similar results have been obtained in actual traffic simulation for networks with exponential degree distribution [62].

To summarize, the above simple weighted RW model for local routing on SF networks indicates that the routing scheme is optimal around the value $\beta^* \approx -1$. At this point, the load is balanced [Eq. (35) and Fig. 4(b)], both the average load and the average packet delivery time are minimum, and the network throughput is maximum (Fig. 6).

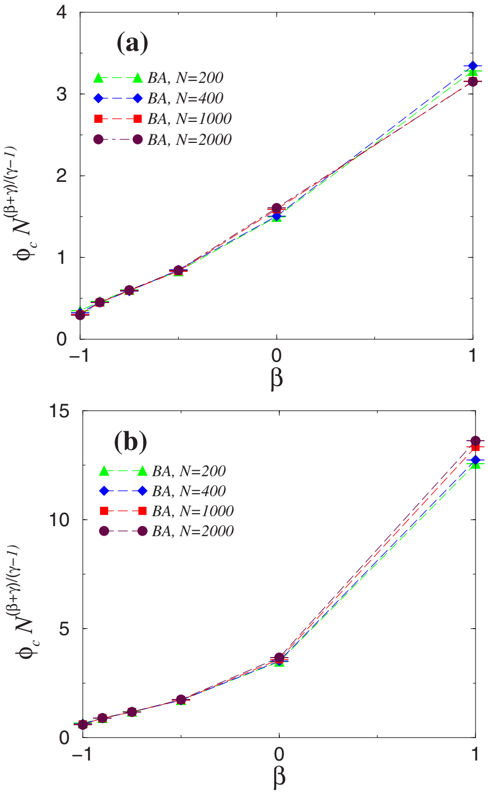

FIG. 7. (Color online) Scaled critical network throughput per node for different system sizes [as suggested by Eq. (43)] for BA networks ($\gamma=3$) as a function of the weighting parameter β in the $\beta>-1$ region for (a) $m=3$ and (b) $m=10$.

From a viewpoint of network vulnerability [99–103], the above results for the weighted RW routing scheme also imply the following. Network failures are often triggered by large load fluctuations at a specific node, then subsequently cascading through the system [102]. Consider a "normal" operating scenario (i.e., failure is *not* due to intentional or targeted attacks), where one gradually increases the packet creation rate ϕ and the overloaded nodes (ones with the highest betweenness) gradually removed from the network [103]. For $\beta>\beta^*\approx-1$ (including the unweighted RW with $\beta=0$), these nodes are the ones with the highest degrees, but uncorrelated SF networks are structurally vulnerable to removing the hubs. At the optimal value of β, not only the network throughput is maximal, and the average packet delivery time is minimal, but the load is balanced: overloads are essentially equally likely to occur at any node and the underlying SF structure is rather resilient to random node removal [99,100]. Thus, at the optimal value of β, the local weighted RW routing simultaneously optimizes network performance and makes the network less vulnerable against inherent system failures due to congestions at the processing nodes.

IV. SUMMARY

We studied the EW process, a prototypical synchronization problem in noisy environments, on weighted uncorrelated scale-free networks. We considered a specific form of the weights, where the strength (and the associated cost) of a link is proportional to $(k_ik_j)^\beta$ with k_i and k_j being the degrees of the nodes connected by the link. Subject to the constraint that the total network cost is fixed, we found that in the mean-field approximation on uncorrelated scale-free graphs, synchronization is optimal at $\beta^*=-1$. Numerical results, based on exact numerical diagonalization of the corresponding network Laplacian on BA SF networks, confirmed the mean-field results, with small corrections to the optimal value of β^*. Although here, because of the presence of noise and the cost constraint, the setup of the problem is quite different, our results are very similar to that of the synchronization of coupled nonlinear oscillators by Zhou *et al.* [28].

Employing our recent connections [36] between the EW process and resistor networks, and some well-known connections between random walks and resistor networks [68–70,91,92], we also explored a naturally related problem of weighted random walks. For the simple toy problem, we found that using the associated RW transition probabilities proportional to a power β of the degree of the neighbors, $P_{ij}\propto A_{ij}k_j^\beta$ [Eq. (31)], the local "routing" is optimal when the $\beta^*=-1$ (in the mean-field approximation). At this optimal network operation point, the load is balanced, both the average load and the average packet delivery time are minimum, and the network throughput is maximum. Since the load is balanced, and thus can lead to local overloads and subsequent failures at any nodes with roughly equal probabilities, the above optimal operating point is also the most resilient one for the underlying scale-free communication network.

Also, while the above local weighted "routing" is overly simplified, some aspects of it can be possibly combined with existing realistic protocols to optimize performance in queue-limited communication networks. For example, existing protocols often utilize an appropriately defined metric for each node, capturing their "distance" (the number of hops) to the current target [104,105]. A node then forwards the packet to a neighbor, which is closer to the target than itself. There may be many nodes satisfying this criterion, so the forwarding node could employ the weighting RW scheme [Eq. (31)], applied to this subset, to select the next node. This may result in improved delivery times and in the delaying of the onset of congestion.

Another major ingredient (and simplification) of the prototypical transport problems considered in this work was that the source and target role of the nodes (e.g., packet creation and annihilation rates) were *homogeneous*. That is, despite the degree distribution being heterogeneous, the probability for a node to serve as a source or target in the resistor network (Sec. III A) or, analogously, the packet creation rate and the probability of a node to be a target were *identical* for all nodes (Sec. III B 3). This was a key technical reason why the actual value of the optimal weight exponent turned out to be around $\beta^*\approx-1$. But in more realistic network-transport scenarios, hubs not only have significantly higher degree, but also have a significantly higher rate of "units" (e.g., passengers or packets) entering or leaving the network [43,55]. Even within our simple models (resistor networks or RW routing), preliminary results [106] show that the appropriately "reweighted" average flow (based on the nodes' individual source and target frequency), again, can be maximized. Further, for sufficiently strong heterogeneity in the nodes' individual source and target importance (correlated with the degree), the value of the optimal weight exponent β^* is positive (to optimally support a higher net in and out flux at those nodes), matching the sign of that of empirically observed networks [43,42,55]. Further research is in progress along this direction [106].

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